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ORIGINAL PAPER

A lower bound on the number of elementary components of essentially disconnected generalized polyomino graphs

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Abstract An essentially disconnected generalized polyomino graph is defined as a generalized polyomino graph with some perfect matchings and forbidden edges. The number of perfect matchings of a generalized polyomino graph G is the product of the number of perfect matchings of each elementary component in G. In this paper, we obtain a lower bound on the number of elementary components of essentially disconnected generalized polyomino graphs.

Keywords Generalized polyomino graph · Elementary component · Perfect matching · Essentially disconnected · Lower bound

1 Introduction

A *polyomino graph* [1,19], also called chessboards [3] or square-cell configurations (lattice animals) [7,8,18], is a finite 2-connected geometric graph in which every interior face is bounded by a regular square of side length 1 (i.e. called a cell). Polyomino graphs have attracted some mathematicians' considerable attentions, for many interesting combinatorial subjects are yielded from them, such as hypergraphs [1], domination problem [3,6], rook polyominal [13], etc. A *generalized polyomino graph* G [5] can be obtained from a polyomino graph H by deleting all the vertices and edges in the interior of a group of pairwise disjoint cycles $C_1, C_2, \ldots, C_k$ ($k \geqslant 1$) which are inside H, i.e. $C_i (i = 1, 2, \ldots, k)$ contains no vertex on the perimeter of H. These cycles are called the inner perimeters of G, while the perimeter C_0 of H is called the outer perimeter of G. For a generalized polyomino graph G, the *characteristic graph* $C(G)$ is defined as the graph whose vertex set is the set of the interior face of

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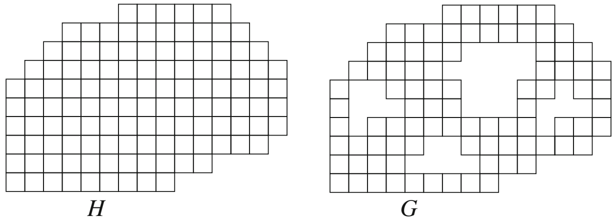


Fig. 1 A polyomino graph H and a generalized polyomino graph G from H

G bounded by a square of side length 1, and two vertices of $C(G)$ are adjacent iff the corresponding interior faces have an edge in common. A generalized polyomino graph G and a polyomino graph H from which G is obtained are given in Fig. 1.

A *perfect matching of a graph* G is a set of independent edges of G covering all vertices of G. An edge of a polyomino graph or a generalized polyomino graph G with at least one perfect matching is said to be a *forbidden single (double)* edge if it belongs to none (all) of the perfect matchings of G and *allowed* otherwise. An edge is said to be a *forbidden* edge if it is either a forbidden single edge or a forbidden double edge. A polyomino graph or generalized polyomino graph G is said to be *elementary* if it has no forbidden edge. Otherwise, it is said to be *essentially disconnected* [4].

The perfect matchings of polyomino graphs and generalized polyomino graphs have been studied widely. For polyomino graphs, the perfect matching problem is closely related to the dimer problem in crystal physics [9,10,14]. John et al. [9] and Sachs [16] also considered the enumeration of its perfect matchings. In addition, Berge et al. [1] studied the generalized covering problem for polyomino graphs by introducing hypergraphs. Zhang [19] gave the necessary and sufficient conditions to have a perfect matching. For a generalized polyomino graph, it is not difficult to see that a domino tiling of it corresponds to a perfect matching of its characteristic graph. Thus some domino tiling problems are reduced to the enumeration problem of perfect matchings of generalized polyomino graphs. In [15] several explicit expressions for the number of perfect matchings for some special types of generalized polyomino graphs were given. Chen [5] also obtained the necessary and sufficient conditions to have a perfect matching.

But it is well known that the enumerating of perfect matchings of a graph is NP-hard [11]. Note that the number of perfect matchings of a polyomino graph or generalized polyomino graph G is the product of the number of perfect matchings of each elementary component in G [5,19]. A natural problem is how many elementary components a graph can be decomposed into. That is why we consider the lower bound on the number of elementary components of polyomino graphs or generalized polyomino graphs. For polyomino graphs, Wei and Ke [17] studied that an essentially disconnected polyomino graph has at least two elementary components and if one of its elementary components is an unit square, then it has at least three elementary components. Furthermore, Liu and Chen [12] proved that an essentially disconnected polyomino graph with an elementary component without vertex on the perimeter has at least five elementary components and constructed essentially disconnected polyomino graphs with 2 or 3 elementary components.

For essentially disconnected generalized polyomino graphs, no more results about the structure feature have been known except that there are necessary and sufficient conditions to have a perfect matching [5]. In this paper, we concentrate ourselves on essentially disconnected generalized polyomino graphs and obtain a lower bound on the number of their elementary components. As an application, we can decompose an essentially disconnected generalized polyomino graph G into a number of elementary components such that the number of perfect matchings of G is equal to the product of those of its components.

2 Definitions and notations

Let G be a generalized polyomino graph with perimeters $C_0, C_1, \ldots, C_k$, where C_0 is the outer perimeter of G (i.e. the perimeter of the corresponding polyomino graph), and $C_1, C_2, \ldots, C_k$ is the inner perimeters of G (i.e. the perimeters of the holes). The following concept of special edge cut and standard combination plays an important role in our investigations.

Definition 2.1 ([5]) A straight line segment P_1P_2 is called a *cut segment* from C_i to C_j if

1. P_1 is the center of an edge e_1 on some perimeter C_i and P_2 is the center of an edge e_2 on some perimeter C_j;
2. P_1P_2 and all edges of G form an angle of $\pi/4$;
3. any point of P_1P_2 is either an interior or a perimeter point of some square of G.

Definition 2.2 ([5]) A broken line segment P_1QP_2 is called a *generalized cut segment* (g-cut segment) from C_i to C_j if

1. P_1 is the center of an edge e_1 on some perimeter C_i and P_2 is the center of an edge e_2 on some perimeter C_j;
2. P_1Q and P_2Q form an angle of $\pi/2$;
3. Q is the center of some edge e which is the bisector of the right angle $\angle P_1QP_2$;
4. any point of P_1QP_2 is either an interior or a perimeter point of some square of G.

Definition 2.3 ([5]) A *special cut segment* is either a cut segment or a g-cut segment. A *special edge cut* R is the set of edges of G intersected by a special cut segment from C_i to C_j, denoted by E_{ij}.

The cut segments $P_{1a}P_{2a}$, $P_{1b}P_{2b}$, $P_{1c}P_{2c}$, $P_{1e}P_{2e}$ and g-cut segment $P_{1d}Q_dP_{2d}$ are shown as in Fig. 2.

It is obvious that two special edge cuts are disjoint if their corresponding special cut segments are disjoint. A special edge cut E_{ij} is said to be of *type I* if $i = j$; otherwise, E_{ij} is said to be of *type II*.

It is easy to check that generalized polyomino graphs are bipartite graph. Thus they are 2-colorable. In the following, we make the convention that all the vertices of a generalized polyomino graph G in question have been colored black or white such that any two adjacent vertices of G have different colors. We denote the sets of white and black vertices of G by $W(G)$ and $B(G)$, respectively. Let E be a subset of the

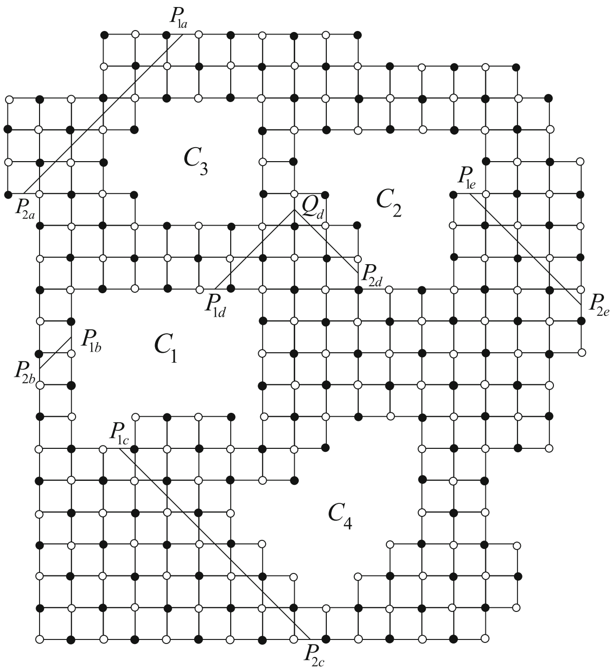


Fig. 2 A generalized polyomino graph G and five special edge cuts in G

edge set of $G.G - E$ is the subgraph of G obtained by deleting all the edges of E. It is evident that $G - E$ has exactly two components if E is a special edge cut of type I, and the end vertices of the edges of E have the same color when they lie in the same component of $G - E$. If E is a special edge cut of type II, then $G - E$ is still connected.

Definition 2.4 Suppose that $E_{i_1i_2}, E_{i_2i_3}, \ldots, E_{i_{t-1}i_t}, E_{i_ti_1}$ are t disjoint special edge cuts of type II, where $E_{i_ji_l}$ corresponds to a special cut segment from C_{i_j} to C_{i_l} and $i_u \neq i_v$ if $u \neq v$. Let $E = E_{i_1i_2} \cup E_{i_2i_3} \cup \cdots \cup E_{i_{t-1}i_t} \cup E_{i_ti_1}.E$ is said to be a standard combination if the end vertices of the edges of E have the same color when they lie in the same component of $G - E$.

In Fig. 2, let E^*_{01} be the special edge cut corresponding to the special cut segment $P_{1b}P_{2b}$, E_{01} the special edge cut corresponding to the special cut segment $P_{1c}P_{2c}$, E_{12} the special edge cut corresponding to the special edge cut segment $P_{1d}Q_dP_{2d}$, E_{20} the special edge cut corresponding to the special edge cut segment $P_{1e}P_{2e}$. Then $E = E_{01} \cup E_{12} \cup E_{20}$ is a standard combination. While the two special edge cuts E_{12}, E_{20} and E^*_{01} do not constitute a standard combination.

In [5], a necessary and sufficient condition for a generalized polyomino graph with some perfect matchings to be essentially disconnected was given.

Theorem 2.5 ([5]) *Let G be a generalized polyomino graph with at least one perfect matching, C_0 the outer perimeter of $G, C_1, C_2, \ldots, C_k$ $(k \geqslant 1)$ the perimeters of* *holes of G. Then G is essentially disconnected if and only if G possesses a special edge cut E_1 of type I, or a standard combination E_2 of type II, satisfying*

i. $|B(G_1)| = |W(G_1)|$ *and* $|B(G_2)| = |W(G_2)|$, *where* $G_i (i = 1, 2)$ *are the two components of* $G - E_1$ *or* $G - E_2$;
ii. *all the edges of* E_1 *or* E_2 *are forbidden single edges.*

The above Theorem implies that for an essentially disconnected generalized polyomino graph G, deleting the forbidden edges which form a special edge cut E_1 of type I or a standard combination E_2 of type II, the subgraph $G - E_1$ or $G - E_2$ is not connected and has at least two connected components.

3 Main results

In this section we prove that for each component $G_i (i = 1, 2)$ of $G - E_1$ or $G - E_2$ which are obtained from Theorem 2.5, there exist some allowed edges, which implies that G_i is elementary or contains an elementary subgraph.

Let G be a generalized polyomino graph, V be a set of vertices of $G. G/V$ denotes the subgraph obtained by deleting all the vertices of V together with their incident edges. For a perfect matching M of G, an *M-alternating cycle* is a cycle whose edges are alternate in M and $E(G) - M$, where $E(G)$ is the edge set of G.

Lemma 3.1 *Let G be a generalized polyomino graph, C_0 the outer perimeter of $G, C_1, C_2, \ldots, C_k$ the inner perimeters of G. Let $v_1, \ldots, v_t$ be t vertices simultaneously on some perimeter C_x of G, $V = \{v_1, \ldots, v_t\}$. Suppose that in G/V, the perimeter C_x of G is broken into t segments with even lengths (i.e. odd vertices). If G/V has a perfect matching M, then G/V has an M-alternating cycle.*

Proof Let G be a generalized polyomino graph with n vertices, m edges, k holes and s squares. We may further assume that G has p external edges (i.e. the edges lying on the perimeter of G), then G has $m - p$ internal edges (i.e. the edges not lying on the perimeter of G). Since each internal edge belongs to two squares, we have $4s = 2(m - p) + p$, i.e.

$$m = 2s + \frac{p}{2}. \tag{1}$$

By Euler's formula [2] which says that for a connected plane graph, the number of vertices plus the number of faces is equal to the number of edges plus two, we have $n + (s + k + 1) = m + 2$, i.e.

$$n - m + s = 1 - k. \tag{2}$$

which together with (1) yields

$$n - s - \frac{p}{2} = 1 - k. \tag{3}$$

On the other hand, suppose that the perfect matching M of G/V contains r external edges of G and hence has $\frac{n-t}{2} - r$ internal edges of G. By the assumption, in G/V the perimeter C_x of G is broken into t segments each of which contains even number edges. Let r_i be the number of edges on perimeter C_i which are contained in M, p_i the number of edges on perimeter C_i. Therefore, we have: $r = r_0+r_1+r_2+\cdots+r_k$, $p = p_0 + p_1 + p_2 + \cdots + p_k$, $r_x \leq \frac{p_x-2t}{2}$ and $r_j \leq \frac{p_j}{2}$ $(j \neq x, 0 \leq j \leq k)$. If some of the perimeters $C_0, C_1, \ldots, C_k$ is an M-alternating cycle, then there is nothing to prove. Now suppose that none of the perimeters $C_0, C_1, \ldots, C_k$ is an M-alternating cycle. Thus, we have $r_x \leq \frac{p_x-2t}{2}$ and $r_j \leq \frac{p_j}{2} - 1$ $(j \neq x, 0 \leq j \leq k)$. Hence,

$$r = r_0 + r_1 + r_2 + \cdots + r_k \leq \frac{p_x - 2t}{2} + \sum_{j \neq x} \left(\frac{p_j}{2} - 1\right) =$$

$$\sum_{j=0}^{k} \left(\frac{p_j}{2}\right) - t - k = \frac{p}{2} - s - k,$$

i.e.

$$r \leq \frac{p}{2} - s - k \tag{4}$$

If none of the squares of G/V is an M-alternating cycle, then at most one edge of each square of G belongs to M. Hence we have $s \geq r + 2(\frac{n-t}{2} - r)$, i.e.

$$s \geq n - r - t. \tag{5}$$

Bearing in mind the inequality (4), we obtain: $s \geqslant n - \frac{p}{2} + k$, i.e.

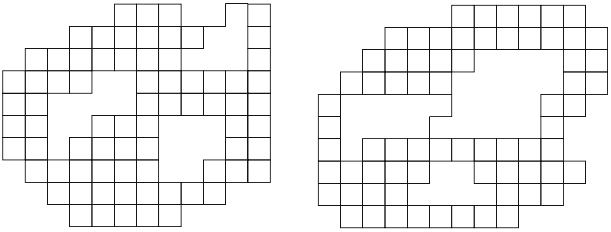
$$n - s - \frac{p}{2} \leqslant -k. \tag{6}$$

Formula (6) is evidently in contradiction with formula (3). The contradiction implies that the assumption about the non-existence of M-alternating cycle which is a square is false. The proof is completed. □

Analogous to the proof of Lemma 3.1, one can easily obtain the same result to polyomino graphs. In other words, if one puts $k = 0$ in the proof of Lemma 3.1, one can reach the conclusion for polyomino graphs.

Lemma 3.2 ([17]) *Let G be a polyomino graph, C the perimeter of G, $v_1, \ldots, v_t$ be t vertices on the perimeter C of G, $V = \{v_1, \ldots, v_t\}$. Suppose that in G/V, the perimeter C of G is broken into t segments with even lengths (i.e. odd vertices). If G/V has a perfect matching M, then G/V has an M-alternating cycle.*

In the following, we introduce a special class of graphs called non-complete generalized polyomino graph. A *non-complete generalized polyomino graph G* is a subgraph of a generalized polyomino graph H and has at least one edge not belonging to any



(a) **(b)**

Fig. 3 Two non-complete generalized polyomino graphs

square of G. Two non-complete generalized polyomino graphs G are shown as in Fig. 3.

Lemma 3.3 *Let G be a non-complete generalized polyomino graph, C_0 the outer perimeter of G, $C_1, C_2, \ldots, C_k$ the inner perimeters of G. Let $v_1, \ldots, v_t$ be t vertices simultaneously on some perimeter C_x of G, $V = \{v_1, \ldots, v_t\}$. Suppose that in G/V, the perimeter C_x of G is broken into t segments with even lengths (i.e. odd vertices). If G/V has a perfect matching M, then G/V has an M-alternating cycle.*

The proof of the above conclusion is fully analogous to that of Lemma 3.1. Hence we omit the details.

By Lemmas 3.1–3.3, if we put $V = \emptyset$, i.e. $t = 0$ in the proof of the Lemmas, we immediately have the following result.

Lemma 3.4 *Let G be a polyomino graph, or a generalized polyomino graph, or a non-complete generalized polyomino graph with at least one perfect matching, then G has an M-alternating cycle.*

We now immediately obtain the following main result.

Theorem 3.5 *If G is an essentially disconnected generalized polyomino graph, then the subgraph obtained from G by deleting all the forbidden single edges and all the end vertices of the forbidden double edges is disconnected.*

Proof By Theorem 2.5, G has a special edge cut E_1 of type I or a standard combination E_2 of type II such that the edges of E_1 or E_2 are forbidden single edges. Then after deleting all the forbidden single edges of E_1 or E_2, G has at least two connected components G_1 and G_2. Each of them may be a component with or without some pendent edges. In the following, we prove that each component $G_i (i = 1, 2)$ has some allowed edges, i.e. G_i has an elementary component which is also an elementary component of G. We distinguish two cases:

Case1. Suppose that G_i has no pendent edge. Then G_i is itself a polyomino graph, or a generalized polyomino graph, or a non-complete generalized polyomino graph. Thus by Lemma 3.4, G_i has some allowed edges (note that all the edges on an M-alternating cycle are allowed edges). Thus, after deleting all the forbidden single edges and all the end vertices of the forbidden double edges, G_i

has a component consisting of allowed edges, i.e. an elementary component. It is clear that this elementary component is also an elementary component of G and is an elementary polyomino graph or an elementary generalized polyomino graph, or a non-complete generalized polyomino graph.

Case 2. Suppose that G_i has some pendent edges, say $u_{ij}v_{ij}(j = 1, 2, \ldots, s)$, where u_{ij} is a vertex of degree 1 in G_i. Since G is a generalized polyomino graph with perfect matchings and all the edges of E_1 or E_2 are forbidden single edges, all the pendent edges $u_{ij}v_{ij}(j = 1, 2, \ldots, s)$ of G_i are forbidden double edges. By deleting all the pendent edges $u_{ij}v_{ij}(j = 1, 2, \ldots, s)$ together with the end vertices u_{ij}, we obtain a polyomino graph, or a generalized polyomino graph, or a non-complete generalized polyomino graph G_i^*. Put $V_i = \{v_{i1}, v_{i2}, \ldots, v_{is}\}$. Then G_i^*/V_i has a perfect matching $M_i - \{u_{ij}, v_{ij}\}$, where M_i is a perfect matching of G_i. Keep in mind the definition of special edge cut of type I and the standard combination of type II, one can check that V_i satisfies the conditions in Lemmas 3.1–3.4. Therefore, G_i^*/V_i has some allowed edges. Consequently, G_i^*/V_i has at least an elementary component which is also an elementary component of G and is an elementary polyomino graph or an elementary generalized polyomino graph, or an elementary non-complete generalized polyomino graph.

Therefore, we come to the conclusion that G has at least two elementary components, one from G_1, and the other from G_2. Each of them may be an elementary polyomino graph, or an elementary generalized polyomino graph, or an elementary non-complete generalized polyomino graph. □

Corollary 3.6 *Let G be an essentially disconnected generalized polyomino graph. Then G has at least two elementary components.*

As Fig. 4 shows that one generalized polyomino graph with two elementary components and the other with five elementary components, where each elementary component is shaded.

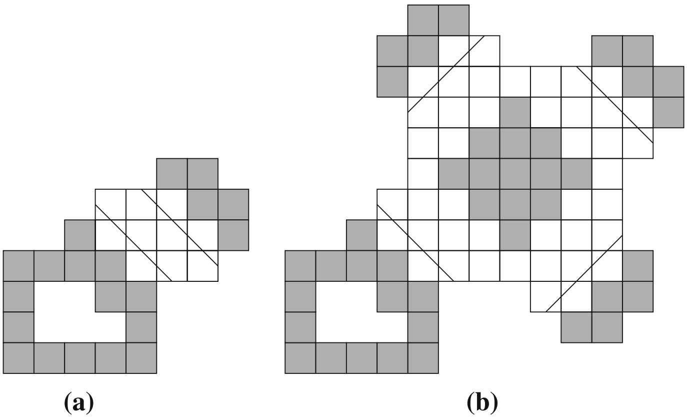


Fig. 4 Two generalized polyomino graphs with two and five elementary components

If G is an essentially disconnected generalized polyomino graph, we can delete all the forbidden edges of G and find that the remained subgraph of G consists of isolated edges and subpolyomino graphs which are elementary components. Therefore, we have the following decomposition theorem.

Theorem 3.7 ([5]) *If G is an essentially disconnected generalized polyomino graph. Denote by $\Phi(G)$ the number of perfect matchings of G. Then*

$$\Phi(G) = \prod_{i=1}^{t} \Phi(G_i)$$

where $\Phi(G_i)(i = 1, 2, \ldots, t)$ are the elementary components of G.

Remark 3.8 Let G be an essentially disconnected generalized polyomino graph. If the number of perfect matchings of each elementary component of G has been given, then one can easily obtain the number of perfect matchings of G by Corollary 3.6 and Theorem 3.7.

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